

Practical bounds for a Dehn parental test

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Abstract

In their article “The shape of hyperbolic Dehn surgery space,” Hodgson and Kerckhoff proved a powerful theorem, half of which they used to make Thurston’s Dehn surgery theorem effective. The calculations derived here use both halves of Hodgson and Kerckhoff’s theorem to give bounds leading towards a practical algorithm to tell, given two orientable hyperbolic 3-manifolds M, N of finite volume, whether or not M is a Dehn filling of N .

1 Introduction

Thurston’s Dehn surgery theorem is one of the brightest gems in the crown of modern geometric 3-manifold topology. It runs as follows:

Theorem 1 ([8], 2.6). *Let $L \subset M^3$ be a link such that $M \setminus L$ has a hyperbolic structure.*

There is a finite set S of filling slopes of components of L such that all Dehn fillings of $M \setminus L$ without a slope from S are hyperbolic.

Following the natural order of mathematics, one wishes to quantify this existence result into something more palpable. One way to make this result more concrete is to provide some measure of length for Dehn filling slopes, then to say that all slopes of large length (for a suitable definition of “large”) yield hyperbolic Dehn fillings. Such theorems include the asymptotics of [7], the Thurston-Gromov 2π -theorem proved in [2], and the 6-theorem of [1] and [6].

Now, the 2π - and 6-theorems just use the naïve notion of length of a slope, viz. length of a geodesic representative in an embedded horospherical torus. In [4] however, Hodgson and Kerckhoff defined *normalized length*¹ \hat{L} , which for a single slope c on a horospherical cusp torus T is just

$$\hat{L}(c) = \frac{\text{length}_T(c)}{\sqrt{\text{area}(T)}}.$$

For a more general Dehn filling c , letting c_T be the slope along which c fills T , Hodgson and Kerckhoff define $\hat{L}(c)$ by requiring $\hat{L} \geq 0$ and

$$\frac{1}{\hat{L}(c)^2} = \sum_{T \subset \partial X} \frac{1}{\hat{L}(c_T)^2},$$

Using this definition of length, they proved the following theorem:

¹This definition was anticipated in [7].

Theorem 2 ([4], Thm. 5.11, Cor. 5.13). *Let X be a compact orientable 3-manifold whose interior admits a complete hyperbolic metric of finite volume. Let c be a Dehn filling of X such that $\hat{L}(c) > 7.5832$. Then*

- $X(c)$ (the filled manifold) itself admits a complete hyperbolic metric on its interior;
- $X \approx X(c) \setminus \gamma$, where γ is a simple closed geodesic of $X(c)$ of length at most 0.156012 and admitting an embedded tube of radius at least $\text{artanh}(1/\sqrt{3})$; and
- $\text{volume}(X) - \text{volume}(X(c)) < 0.198$.

After suitably rephrasing this, it seems to give a practical method for determining Dehn filling heritage:

Corollary 3. *Let M, N be orientable 3-manifolds admitting complete hyperbolic metrics of finite volume on their interior. N is a Dehn filling of M if and only if² either*

- N is a Dehn filling of M along a slope of normalized length less than or equal to 7.5832, or
- M is isometric to $N \setminus \gamma$ for a simple closed geodesic of length less than 0.156012.

The collection of slopes of ∂M with normalized length less than 7.5832 is computable, and likewise the length spectrum of N is computable, and SnapPy can drill out curves and determine isometries, so that is that. Right?

Unfortunately not. The problem is in drilling out curves. SnapPy can only drill out simple closed curves in the dual 1-complex of an ideal triangulation. As explained in [5] on page 264, these may or may not be isotopic (or even homotopic) to a given geodesic which one wishes to drill out.

Fortunately, Theorem 2 is a corollary of a much more powerful theorem, Theorem 5, about how much Dehn filling decreases volume. Theorem 2 follows from the upper bounds in Theorem 5, but Theorem 5 contains lower bounds as well. The calculations below lead to the following theorem:

Theorem 4. *Let M, N be orientable 3-manifolds admitting complete hyperbolic metrics of finite volume on their interiors. Let $\Delta V = \text{Vol}(M) - \text{Vol}(N)$.*

N is a Dehn filling of M if and only if either

- N is a Dehn filling of M along a slope of normalized length less than or equal to 7.5832, or
- N has a closed simple geodesic of length less than $2.879 \cdot \Delta V$, and
- N is a Dehn filling of M along a slope of normalized length \hat{L} such that

$$\frac{4.563}{\Delta V} \leq \hat{L}^2 \leq \frac{20.633}{\Delta V}. \quad (1)$$

This puts Dehn filling heritage for hyperbolic manifolds in terms of procedures either already made rigorous or with a reasonable hope of being made rigorous soon, viz. estimates on volume, length spectra, cusp area, slope length, and (to a lesser extent) isometry testing.

²The only-if part is the content of Theorem 2.

2 Rewriting the Hodgson-Kerckhoff Bounds

The stronger theorem alluded to above is

Theorem 5 ([4], 5.12). *Let X, \hat{L} , and c be as in Theorem 2. Let $\Delta V = \text{vol}(X) - \text{vol}(X(c))$. Let ℓ be the length of the geodesic core of the filling. Then*

$$\frac{1}{4} \cdot \int_{\tilde{z}}^1 \frac{H'(z)}{H(z) \cdot (H(z) - \tilde{G}(z))} dz \leq \Delta V, \quad (2)$$

$$\Delta V \leq \frac{1}{4} \cdot \int_{\tilde{z}}^1 \frac{H'(z)}{H(z) \cdot (H(z) + G(z))} dz, \quad (3)$$

and

$$1/H(\tilde{z}) \leq 2\pi \cdot \ell \leq 1/H(\hat{z}), \quad (4)$$

where $H, G, \tilde{G}, \tilde{z}$, and \hat{z} have the following definitions.

Definition 6.

$$\begin{aligned} K &= 3.3957, & h(z) &= \frac{1+z^2}{z \cdot (1-z^2)}, \\ \tilde{g}(z) &= \frac{(1+z^2)^2}{2 \cdot z^3 \cdot (3-z^2)}, & g(z) &= \frac{1+z^2}{2 \cdot z^3}, \\ H &= h/K, & G &= g/K, & \tilde{G} &= \tilde{g}/K, \\ F(z) &= \frac{H'(z)}{H(z) + G(z)} - \frac{1}{1-z} = \frac{h'(z)}{h(z) + g(z)} - \frac{1}{1-z}, \\ \tilde{F}(z) &= \frac{H'(z)}{H(z) - \tilde{G}(z)} - \frac{1}{1-z} = \frac{h'(z)}{h(z) - \tilde{g}(z)} - \frac{1}{1-z}, \\ f(z) &= K \cdot (1-z) \cdot e^{-\Phi(z)}, & \Phi(z) &= \int_1^z F(w) dw, \\ \tilde{f}(z) &= K \cdot (1-z) \cdot e^{-\tilde{\Phi}(z)}, & \tilde{\Phi}(z) &= \int_1^z \tilde{F}(w) dw, \\ f(\hat{z}) &= \frac{(2\pi)^2}{\hat{L}(c)^2}, & \tilde{f}(\tilde{z}) &= \frac{(2\pi)^2}{\tilde{L}(c)^2} \end{aligned}$$

These definitions are from pp. 1079, 1080, and 1088 of [4]. The reader should note that the above theorem has $2\pi \cdot \ell$ in place of \mathcal{A} . This is valid—see, e.g., Corollary 5.13 of [4].

This gives complicated bounds on ΔV and ℓ in terms of \hat{z} and \tilde{z} . We require simple but not necessarily tight upper and lower bounds on ℓ and $\hat{L}(c)$ in terms of ΔV . In a Dehn parental test, the bounds on ℓ will be used most often; the upper bounds on $\hat{L}(c)$ will be used when the volumes of the putative parent and child P and C are close, *and* C has a very short geodesic. For instance, P might be the Whitehead link complement, and C might be a high-order Dehn filling on the $(-2, 3, 8)$ pretzel link complement.

To get these bounds, we will approximate solutions to inequalities (2) and (3) in \tilde{z} and \hat{z} , respectively, for given ΔV .

2.1 Monotonicities

Let

$$LB(z) = \frac{1}{4} \cdot \int_z^1 \frac{H'(w)}{H(w) \cdot (H(w) - \tilde{G}(w))} dw \quad (5)$$

and

$$UB(z) = \frac{1}{4} \cdot \int_z^1 \frac{H'(w)}{H(w) \cdot (H(w) + G(w))} dw. \quad (6)$$

We intend to solve the inequalities by inverting LB and UB . This will work if we know the monotonicity of LB and UB . We will require the monotonicity of several other functions as well, and the (very calculational) proofs are in proof-hint notation. It behooves us then to introduce “ \sim .”

Definition 7. For all real x and y , $x \sim y$ when $\text{sgn}(x) = \text{sgn}(y)$, where $\text{sgn}(x)$ is the signum function $\text{sgn}(0) = 0$, else $\text{sgn}(x) = |x|/x$.

Lemma 8. LB is decreasing on $(\sqrt{\sqrt{5}-2}, 1)$.

Lemma 9. UB is decreasing on $(\sqrt{\sqrt{5}-2}, \infty)$.

Proof of Lemma 8.

$$\begin{aligned} & LB'(z) \\ = & \{ \text{by definition of } LB \} \\ & -\frac{1}{4} \cdot \frac{H'(z)}{H(z) \cdot (H(z) - \tilde{G}(z))} \\ = & \{ \text{algebra} \} \\ & -K/4 \cdot \frac{h'(z)}{h(z) \cdot (h(z) - \tilde{g}(z))} \\ \sim & \{ K > 0 \} \\ & -\frac{h'(z)}{h(z) \cdot (h(z) - \tilde{g}(z))} \\ = & \{ \text{algebra} \} \\ & -\frac{z^2 \cdot (z^2 - 3) \cdot (z^4 + 4 \cdot z^2 - 1)}{(z^2 + 1)^2 \cdot (z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\ \sim & \{ z > \sqrt{\sqrt{5}-2} \Rightarrow z^2 - 3 < 0 \} \\ & \frac{z^2 \cdot (z^4 + 4 \cdot z^2 - 1)}{(z^2 + 1)^2 \cdot (z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\ \sim & \{ z > \sqrt{\sqrt{5}-2} \Rightarrow z^2 / (z^2 + 1)^2 > 0 \} \\ & \frac{z^4 + 4 \cdot z^2 - 1}{(z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\ \sim & \{ z > \sqrt{\sqrt{5}-2} \Rightarrow z^4 + 4 \cdot z^2 - 1 > 0 \} \end{aligned}$$

$$\begin{aligned}
& \frac{1}{(z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
\sim & \left\{ \begin{aligned} z &> \sqrt{\sqrt{5} - 2} \Rightarrow z > \sqrt{2} - 1, \\ z &> \sqrt{2} - 1 \Rightarrow z^2 + 2 \cdot z - 1 > 0 \end{aligned} \right\} \\
& \frac{1}{z^2 - 2 \cdot z - 1} \\
\sim & \left\{ \begin{aligned} z &> \sqrt{\sqrt{5} - 2} \Rightarrow z > 1 - \sqrt{2}, \\ z &< 1 \Rightarrow z < 1 + \sqrt{2}, \\ 1 - \sqrt{2} &< z < 1 + \sqrt{2} \Rightarrow z^2 - 2 \cdot z - 1 < 0 \end{aligned} \right\} \\
& -1.
\end{aligned}$$

By calculus, therefore, LB is decreasing on $(\sqrt{\sqrt{5} - 2}, 1)$. \square

Proof of Lemma 9.

$$\begin{aligned}
& UB'(z) \\
= & \quad \{\text{by definition of } UB\} \\
& - \frac{1}{4} \cdot \frac{H'(z)}{H(z) \cdot (H(z) + G(z))} \\
= & \quad \{\text{algebra}\} \\
& - \frac{K}{4} \cdot \frac{h'(z)}{h(z) \cdot (h(z) + g(z))} \\
= & \quad \{\text{more algebra}\} \\
& - \frac{K}{2} \cdot \frac{z^2 \cdot (z^4 + 4 \cdot z^2 - 1)}{(z^2 + 1)^3} \\
\sim & \quad \{K > 0; \quad z \neq 0\} \\
& - (z^4 + 4 \cdot z^2 - 1) \\
\sim & \quad \left\{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z^4 + 4 \cdot z^2 - 1 > 0 \right\} \\
& -1.
\end{aligned}$$

Again, by calculus, UB is decreasing, on $(\sqrt{\sqrt{5} - 2}, \infty)$. \square

Therefore, the first two inequalities of Theorem 5 are equivalent, respectively, to $\tilde{z} \geq LB^{-1}(\Delta V)$ and $UB^{-1}(\Delta V) \geq \hat{z}$.

Next, we should do the same to the inequalities (4), getting bounds for \tilde{z} and \hat{z} in terms of ℓ . To do that we need H 's monotonicity. We can then play the various inequalities off one another to get our desired result. Also, we should determine the monotonicities of f and \tilde{f} ; they will prove useful later.

Lemma 10. H is increasing on $(\sqrt{\sqrt{5} - 2}, \infty)$.

Lemma 11. f is decreasing on $(\sqrt{\sqrt{5} - 2}, \infty)$.

Lemma 12. \tilde{f} is decreasing on $(\sqrt{\sqrt{5}-2}, \sqrt{3})$.

Proof of Lemma 10.

$$\begin{aligned}
& H'(z) \\
&= \quad \{\text{by Def}\} \\
& \quad h'(z)/K \\
&\sim \quad \{K > 0\} \\
& \quad h'(z) \\
&= \quad \{\text{calculus}\} \\
& \quad \frac{z^4 + 4 \cdot z - 1}{(z-1)^2 \cdot z^2 \cdot (z+1)^2} \\
&\sim \\
& \quad z^4 + 4 \cdot z - 1 \\
&\sim \quad \left\{ z > \sqrt{\sqrt{5}-2} \Rightarrow z^4 + 4 \cdot z - 1 > 0 \right\} \\
& \quad 1.
\end{aligned}$$

By calculus, H is increasing if $z > \sqrt{\sqrt{5}-2}$ —in particular, if $z \in (\sqrt{\sqrt{5}-2}, 1)$. \square

Proof of Lemma 11.

$$\begin{aligned}
& f'(z) \\
&= \quad \{\text{calculus, algebra}\} \\
& \quad K \cdot e^{-\Phi(z)} \cdot (-1 + (1-z) \cdot (-\Phi'(z))) \\
&\sim \quad \left\{ K > 0; \quad e^{-\Phi(z)} > 0 \right\} \\
& \quad (z-1) \cdot \Phi'(z) - 1 \\
&= \quad \{\text{fund. thm. of calculus}\} \\
& \quad (z-1) \cdot F(z) - 1 \\
&= \quad \{\text{algebra}\} \\
& \quad - \frac{2 \cdot z \cdot (z^4 + 4 \cdot z^2 - 1)}{(z+1) \cdot (z^2+1)^2} \\
&\sim \\
& \quad - (z^4 + 4 \cdot z^2 - 1) \\
&\sim \quad \left\{ z > \sqrt{\sqrt{5}-2} \Rightarrow z^4 + 4 \cdot z - 1 > 0 \right\} \\
& \quad -1.
\end{aligned}$$

By calculus, f is decreasing if $z > \sqrt{\sqrt{5}-2}$ —in particular, if $z \in (\sqrt{\sqrt{5}-2}, 1)$. \square

Proof of Lemma 12.

$$\begin{aligned}
& \tilde{f}'(z) \\
&= \quad \{\text{calculus, algebra}\} \\
& \quad K \cdot e^{-\tilde{\Phi}(z)} \cdot \left(-1 + (1-z) \cdot (-\tilde{F}(z)) \right) \\
&\sim \quad \left\{ K > 0; \quad e^{-\tilde{\Phi}(z)} > 0 \right\} \\
& \quad (z-1) \cdot \tilde{F}(z) - 1 \\
&= \quad \{\text{algebra}\} \\
& \quad \frac{-2 \cdot z \cdot (z^2 - 3) \cdot (z^4 + 4 \cdot z - 1)}{(z+1) \cdot (z^2 + 1) \cdot (z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \quad \left\{ z \in \left(\sqrt{\sqrt{5} - 2}, \sqrt{3} \right) \Rightarrow z^2 - 3 < 0 \right\} \\
& \quad \frac{2 \cdot z \cdot (z^4 + 4 \cdot z - 1)}{(z+1) \cdot (z^2 + 1) \cdot (z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \quad \{ z > \sqrt{\sqrt{5} - 2} \Rightarrow z > 0 \} \\
& \quad \frac{z^4 + 4 \cdot z - 1}{(z^2 - 2 \cdot z - 1) \cdot (z^2 + 2 \cdot z - 1)} \\
&\sim \quad \{\text{latter half of Lemma 8} \} \\
& \quad -1.
\end{aligned}$$

□

2.2 Complicated upper bound on ℓ

Plainly we already have an upper bound on ℓ , viz. $\ell \leq 1/(2\pi \cdot H(\hat{z}))$. We just need to put the right-hand side in terms of ΔV .

In fact, since H is increasing, $1/(2\pi \cdot H)$ is decreasing. Therefore we just need a lower bound on \hat{z} ; applying $1/(2\pi \cdot H)$ to this lower bound will give us a bound on ℓ .

At this point, one could use the standing assumption in [4] after p. 1079 that all variables named z represent $\tanh \rho$ for some $\rho > \text{artanh}(1/\sqrt{3})$. Therefore, $\hat{z} > \sqrt{1/3}$. As a matter of fact, this is where the bounds in Theorem 2 come from. But we would like a better bound for small ΔV .

Now, $UB(\hat{z}) \geq \Delta V$. Unfortunately UB is decreasing, so this doesn't give a lower bound on \hat{z} . Also, \hat{z} is defined by $f(\hat{z}) = (2\pi)^2/(\hat{L}(c)^2)$, but all we know about $\hat{L}(c)$ is $\hat{L}(c) > 7.5832$. In fact, this bound is taken from the standing assumption on z .

However, we also know $f(\hat{z}) = \tilde{f}(\tilde{z})$ and \tilde{f} both are decreasing. Therefore, if we can get a lower bound on \tilde{z} , we get a lower bound on \hat{z} , via upper bounds on $f(\hat{z}) = \tilde{f}(\tilde{z})$.

Finally, (2) from Theorem 5 says $LB(\tilde{z}) \leq \Delta V$, and LB is decreasing on $(\sqrt{\sqrt{5} - 2}, 1)$. $\sqrt{1/3} > \sqrt{\sqrt{5} - 2}$, so this yields a lower bound on \tilde{z} , and hence an upper bound on ℓ , in terms of ΔV ; to wit,

$$\ell \leq \frac{1}{2\pi \cdot (H \circ s \circ \tilde{f} \circ BL)(\Delta V)}, \tag{7}$$

where $s(f(\hat{z})) = \hat{z}$ and $BL(LB(\hat{z})) = \hat{z}$ for $\hat{z}, \hat{z} \in (\sqrt{1/3}, 1)$, and $s : (0, f(\sqrt{1/3})) \rightarrow (\sqrt{1/3}, 1)$, $BL : (0, LB(\sqrt{1/3})) \rightarrow (\sqrt{1/3}, 1)$.

This bound is valid only when ΔV is in the domain of BL . If this is not the case, then the right-hand side should be replaced by Hodgson and Kerckhoff's original bound 0.156012.

2.3 Complicated bounds on $\hat{L}(c)$

We know $\frac{(2\pi)^2}{\hat{L}(c)^2} = f(\hat{z}) = \tilde{f}(\hat{z})$. We just got upper bounds on this, yielding a lower bound for $\hat{L}(c)$. More explicitly,

$$\hat{L}(c)^2 \geq \frac{(2\pi)^2}{\tilde{f}(BL(\Delta V))}. \quad (8)$$

To get an upper bound on $\hat{L}(c)$, we can get a lower bound on $f(\hat{z})$, which would result from an upper bound on \hat{z} (since f is decreasing), which would result from a lower bound on $UB(\hat{z})$ (since UB is decreasing). But $\Delta V \leq UB(\hat{z})$ by assumption. So

$$\hat{L}(c)^2 \leq \frac{(2\pi)^2}{f(BU(\Delta V))}, \quad (9)$$

where $BU : (0, UB(\sqrt{1/3})) \rightarrow (\sqrt{1/3}, 1)$ satisfies $BU(UB(\hat{z})) = \hat{z}$ for $\hat{z} \in (\sqrt{1/3}, 1)$.

2.4 Nice bounds

Since these bounds depend upon inverting functions defined by integrals, one cannot expect a computer to calculate the bounds very quickly. But if we approximate the functions and relax the bounds, we can get decent running times.

The conditions which the approximations should satisfy (in order to accord with (7), (8), and (9)) are not difficult to derive. For instance, an approximation η to $1/(2\pi \cdot H)$ should be decreasing, since $1/(2\pi \cdot H)$ is itself decreasing and we want a reasonable approximation; and η should be greater than $1/(2\pi \cdot H)$ so that we can deduce

$$\ell \leq (\eta \circ s \circ \tilde{f} \circ BL)(\Delta V)$$

from (7). In fact, $\eta(z) = K \cdot (1 - z)/(2\pi)$ suffices. Useful approximations for all the necessary functions are as follows:

Lemma 13.

$$1/h(z) \leq 1 - z, \quad (10)$$

$$f(z) \geq A \cdot (1 - z), \quad (11)$$

$$\tilde{f}(z) \leq B \cdot (1 - z), \quad (12)$$

$$LB(z) \geq C \cdot (1 - z), \quad (13)$$

$$UB(z) \leq D \cdot (1 - z). \quad (14)$$

where

$$\begin{aligned}
A &= K \cdot e^{-\Phi(\sqrt{1/3})}; \\
\tilde{F}(\beta) &= 0, \quad \beta \in (\sqrt{1/3}, 1); \\
B &= K \cdot e^{-\tilde{\Phi}(\beta)}; \\
t &= \frac{h'}{h \cdot (h - \bar{g})}; \\
C &= K \cdot t(\sqrt{1/3})/4; \\
D &= K/4;
\end{aligned}$$

Proof of (10).

$$1 - z - \frac{1}{h(z)} = \frac{(1 - z)^2}{1 + z^2} \geq 0.$$

□

Proof of (11). Assume $z \in (\sqrt{1/3}, 1)$. Now, by definition,

$$F(z) = -\frac{z^4 + 6 \cdot z^2 + 4 \cdot z + 1}{(z + 1) \cdot (z^2 + 1)^2}.$$

But

$$\begin{aligned}
F(z) &= -\frac{z^4 + 6 \cdot z^2 + 4 \cdot z + 1}{(z + 1) \cdot (z^2 + 1)^2} \\
&\Rightarrow \quad \{ \text{algebra} \} \\
&F < 0 \text{ on } (\sqrt{1/3}, 1) \\
&\Rightarrow \quad \{ \text{calculus; } z \in (\sqrt{1/3}, 1) \} \\
&\int_z^1 F(w) dw \geq \int_{\sqrt{1/3}}^1 F(w) dw \\
&\equiv \quad \{ \text{calculus, algebra} \} \\
&\int_1^z F(w) dw \leq \int_1^{\sqrt{1/3}} F(w) dw \\
&\equiv \quad \{ \text{definition of } \Phi \} \\
&\Phi(z) \leq \Phi(\sqrt{1/3}) \\
&\equiv \quad \{ x \mapsto e^{-x} \text{ is decreasing} \} \\
&e^{-\Phi(z)} \geq e^{-\Phi(\sqrt{1/3})} \\
&\equiv \quad \{ z \in (\sqrt{1/3}, 1) \Rightarrow 1 - z > 0; K > 0 \} \\
&K \cdot e^{-\Phi(z)} \cdot (1 - z) \geq K \cdot e^{-\Phi(\sqrt{1/3})} \cdot (1 - z) \\
&\equiv \quad \{ \text{definition of } f \} \\
&f(z) \geq K \cdot e^{-\Phi(\sqrt{1/3})} \cdot (1 - z) \\
&\equiv \quad \{ \text{definition of } A \}
\end{aligned}$$

$$f(z) \geq A \cdot (1 - z)$$

□

Proof of (12). $\tilde{F}(1) = 1$, $\tilde{F}(\sqrt{1/3}) < 0$, and \tilde{F} has exactly one root β in $(\sqrt{1/3}, 1)$. Thus if $z \in (\sqrt{1/3}, 1)$, then

$$\begin{aligned}
& \int_z^1 \tilde{F}(w) dw \leq \int_\beta^1 \tilde{F}(w) dw \\
& \equiv \{ \text{calculus} \} \\
& \int_1^z \tilde{F}(w) dw \geq \int_1^\beta \tilde{F}(w) dw \\
& \equiv \{ \text{definition of } \tilde{\Phi} \} \\
& \tilde{\Phi}(z) \geq \tilde{\Phi}(\beta) \\
& \equiv \{ \text{algebra} \} \\
& -\tilde{\Phi}(z) \leq -\tilde{\Phi}(\beta) \\
& \equiv \{ x \mapsto e^x \text{ is increasing} \} \\
& e^{-\tilde{\Phi}(z)} \leq e^{-\tilde{\Phi}(\beta)} \\
& \equiv \{ \text{algebra} \} \\
& K \cdot (1 - z) \cdot e^{-\tilde{\Phi}(z)} \leq K \cdot (1 - z) \cdot e^{-\tilde{\Phi}(\beta)} \\
& \equiv \{ \text{definition of } \tilde{f} \} \\
& \tilde{f}(z) \leq K \cdot (1 - z) \cdot e^{-\tilde{\Phi}(\beta)} \\
& \equiv \{ \text{definition of } B \} \\
& \tilde{f}(z) \leq B \cdot (1 - z).
\end{aligned}$$

But the initial statement is just equation (2). □

Proof of (13). For variety, we do this proof backwards. We seek a C such that for all $z \in (\sqrt{1/3}, 1)$, $LB(z) \geq C \cdot (1 - z)$:

$$\begin{aligned}
& \langle \forall z : LB(z) \geq C \cdot (1 - z) \rangle \\
& \equiv \{ \text{let } lb(z) = \int_z^1 h'/(h \cdot (h - \tilde{g})) \} \\
& \langle \forall z : K \cdot lb(z)/4 \geq C \cdot (1 - z) \rangle \\
& \equiv \{ \text{algebra} \} \\
& \langle \forall z : lb(z) \geq 4 \cdot C \cdot (1 - z)/K \rangle \\
& \Leftarrow \{ \text{calculus} \} \\
& h'/(h \cdot (h - \tilde{g})) \geq 4 \cdot C/K \text{ on } (\sqrt{1/3}, 1).
\end{aligned}$$

In other words, we just need a lower bound on $t = h'/(h \cdot (h - \tilde{g}))$ over $(\sqrt{1/3}, 1)$. Now,

$$t'(z) = \frac{4 \cdot (1 - z) \cdot (z + 1) \cdot p(z)}{(z^2 + 1)^3 \cdot (z^2 - 2 \cdot z - 1)^2 \cdot (z^2 + 2 \cdot z - 1)^2},$$

where

$$p(z) = 5 \cdot z^8 - 6 \cdot z^6 + 88 \cdot z^4 - 26 \cdot z^2 + 3.$$

It is clear that on $(\sqrt{1/3}, 1)$, $t' \sim p$. Now,

$$\begin{aligned} &= p(z) \\ &= 5 \cdot z^8 - 6 \cdot z^6 + 2 \cdot z^4 + 86 \cdot z^4 - 26 \cdot z^2 + 3 \\ &= z^4 \cdot (5 \cdot (z^2)^2 - 6 \cdot (z^2) + 2) \\ &+ 86 \cdot (z^2)^2 - 26 \cdot z^2 + 3. \end{aligned}$$

$(-6)^2 - 4 \cdot 5 \cdot 2 < 0$ and $(-26)^2 - 4 \cdot 86 \cdot 3 < 0$. Therefore, $5 \cdot z^2 - 6 \cdot z + 2$ has constant sign, and $86 \cdot z^2 - 26 \cdot z + 3$ does too. By evaluation at 0, this sign is positive on both. Therefore p is positive. That is, $t' > 0$ on $(\sqrt{1/3}, 1)$. Consequently, t achieves its smallest value at $\sqrt{1/3}$. That is, $t \geq t(\sqrt{1/3})$. So we have, finally,

$$\begin{aligned} &\langle \forall z : LB(z) \geq C \cdot (1 - z) \rangle \\ &\Leftarrow \{ \text{see above} \} \\ &C = K \cdot t(\sqrt{1/3})/4. \end{aligned}$$

□

Proof of (14). Likewise, we do this proof backwards. We seek a D such that for all $z \in (\sqrt{1/3}, 1)$, $UB(z) \leq D \cdot (1 - z)$:

$$\begin{aligned} &\langle \forall z : UB(z) \leq D \cdot (1 - z) \rangle \\ &\equiv \{ \text{let } ub(z) = \int_z^1 h'/(h \cdot (h + g)) \} \\ &\langle \forall z : K \cdot ub(z)/4 \leq D \cdot (1 - z) \rangle \\ &\equiv \{ \text{algebra} \} \\ &\langle \forall z : ub(z) \leq 4 \cdot D \cdot (1 - z)/K \rangle \\ &\Leftarrow \{ \text{calculus} \} \\ &h'/(h \cdot (h + g)) \leq 4 \cdot D/K \text{ on } (\sqrt{1/3}, 1). \end{aligned}$$

In other words, we just need an upper bound on $T = h'/(h \cdot (h + g))$ over $(\sqrt{1/3}, 1)$. Now,

$$T'(z) = -\frac{4 \cdot z \cdot (z^4 - 10 \cdot z^2 + 1)}{(z^2 + 1)^4}.$$

Plainly, $T'(z) \sim -z^4 + 10 \cdot z^2 - 1$ on $(\sqrt{1/3}, 1)$. This has four real roots, $\pm\sqrt{5 \pm 2 \cdot \sqrt{6}}$, none of which lies in $(\sqrt{1/3}, 1)$. $-1 + 10 - 1 > 0$, so T' is positive on $(\sqrt{1/3}, 1)$. That is, T is increasing on $(\sqrt{1/3}, 1)$. So it takes its maximum at 1, where its value is just 1! In conclusion, then,

$$\begin{aligned} &\langle \forall z : UB(z) \leq D \cdot (1 - z) \rangle \\ &\Leftarrow \{ \text{see above} \} \\ &D = K/4. \end{aligned}$$

□

Lemma 14.

$$\frac{1}{2\pi \cdot (H \circ s \circ \tilde{f} \circ BL)(\Delta V)} \leq \alpha \cdot \Delta V, \quad (15)$$

$$\frac{(2\pi)^2}{\tilde{f}(BL(\Delta V))} \geq \delta \cdot \frac{1}{\Delta V}, \quad (16)$$

and

$$\frac{(2\pi)^2}{f(BU(\Delta V))} \leq \gamma \cdot \frac{1}{\Delta V}, \quad (17)$$

where

$$\begin{aligned} \alpha &= \frac{2 \cdot e^{\Phi(\sqrt{1/3}) - \tilde{\Phi}(\beta)}}{\pi \cdot t(\sqrt{1/3})}, \\ \delta &= \frac{(2\pi)^2 \cdot e^{\tilde{\Phi}(\beta)} \cdot t(\sqrt{1/3})}{4}, \\ \gamma &= \frac{(2\pi)^2 \cdot e^{\Phi(\sqrt{1/3})}}{4}, \end{aligned}$$

and as above,

$$\tilde{F}(\beta) = 0, \beta \in (\sqrt{1/3}, 1).$$

Proof of (15).

$$\begin{aligned} & 1/(h \circ s \circ \tilde{f} \circ BL)(\Delta V) \\ = & \{ \text{algebra} \} \\ & \left(\frac{1}{h} \circ s \circ \tilde{f} \circ BL \right) (\Delta V) \\ \leq & \{ (10) \} \\ & 1 - (s \circ \tilde{f} \circ BL)(\Delta V) \\ \leq & \{ s, f \text{ inverse; (11)} \} \\ & 1 - (1 - \tilde{f}(BL(\Delta V))/A) \\ = & \{ \text{algebra} \} \\ & \tilde{f}(BL(\Delta V))/A \\ \leq & \{ (12) \} \\ & \frac{B}{A} \cdot (1 - BL(\Delta V)) \\ \leq & \{ BL, LB \text{ inverse; (13)} \} \\ & \frac{B}{A} \cdot (1 - (1 - \Delta V/C)) \\ = & \{ \text{algebra} \} \\ & \frac{B}{A \cdot C} \cdot \Delta V. \end{aligned}$$

Consequently,

$$\begin{aligned} & 1/(2\pi \cdot (H \circ s \circ \tilde{f} \circ BL)(\Delta V)) \\ = & \{ \text{definition of } H \} \end{aligned}$$

$$\begin{aligned}
& K/(2\pi \cdot (h \circ s \circ \tilde{f} \circ BL)(\Delta V)) \\
= & \{ \text{ algebra } \} \\
& \frac{K}{2\pi} \cdot 1/(h \circ s \circ \tilde{f} \circ BL)(\Delta V) \\
\leq & \{ \text{ see above } \} \\
& \frac{K}{2\pi} \cdot \frac{B}{A \cdot C} \cdot \Delta V.
\end{aligned}$$

Finally,

$$\begin{aligned}
& \frac{K}{2\pi} \cdot \frac{B}{A \cdot C} \\
= & \{ \text{ definitions of } A, B, C \} \\
& \frac{K \cdot K \cdot e^{-\tilde{\Phi}(\beta)} \cdot 4}{2\pi \cdot K \cdot e^{-\Phi(\sqrt{1/3})} \cdot K \cdot t(\sqrt{1/3})} \\
= & \{ \text{ algebra } \} \\
& 2 \cdot e^{\Phi(\sqrt{1/3}) - \tilde{\Phi}(\beta)} / (\pi \cdot t(\sqrt{1/3})) \\
= & \{ \text{ definition of } \alpha \} \\
& \alpha.
\end{aligned}$$

□

Proof of (16).

$$\begin{aligned}
& 1/(\tilde{f}(BL(\Delta V))) \\
\geq & \{ (12); \text{ algebra } \} \\
& 1/(B \cdot (1 - BL(\Delta V))) \\
\geq & \{ (13); \text{ algebra } \} \\
& 1/(B \cdot (1 - (1 - \Delta V/C))) \\
= & \{ \text{ algebra } \} \\
& (C/B) \cdot (1/\Delta V)
\end{aligned}$$

Consequently,

$$\frac{(2\pi)^2}{\tilde{f}(BL(\Delta V))} \leq \frac{(2\pi)^2 \cdot C}{B} \cdot \frac{1}{\Delta V}.$$

Finally,

$$\begin{aligned}
& (2\pi)^2 \cdot C/B \\
= & \{ \text{ definitions of } B, C \} \\
& \frac{(2\pi)^2 \cdot K \cdot t(\sqrt{1/3})}{K \cdot e^{-\tilde{\Phi}(\beta)} \cdot 4} \\
= & \{ \text{ algebra } \} \\
& (2\pi)^2 \cdot t(\sqrt{1/3}) \cdot e^{\tilde{\Phi}(\beta)} / 4. \\
= & \{ \text{ definition of } \delta \}
\end{aligned}$$

δ .

□

Proof of (17).

$$\begin{aligned}
& 1/(f(BU(\Delta V))) \\
& \leq \{ (11); \text{ algebra } \} \\
& \quad 1/(A \cdot (1 - BU(\Delta V))) \\
& \leq \{ (14) \} \\
& \quad 1/(A \cdot (1 - (1 - \Delta V/D))) \\
& = \{ \text{ algebra } \} \\
& \quad (D/A) \cdot (1/\Delta V).
\end{aligned}$$

Consequently,

$$\frac{(2\pi)^2}{f(BU(\Delta V))} \leq \frac{(2\pi)^2 \cdot D}{A} \cdot \frac{1}{\Delta V}.$$

Finally,

$$\begin{aligned}
& (2\pi)^2 \cdot D/A \\
& = \{ \text{ definitions of } A, D \} \\
& \quad \frac{(2\pi)^2 \cdot K}{K \cdot e^{-\Phi(\sqrt{1/3})} \cdot 4} \\
& = \{ \text{ algebra } \} \\
& \quad (2\pi)^2 \cdot e^{\Phi(\sqrt{1/3})}/4 \\
& = \{ \text{ definition of } \gamma \} \\
& \quad \gamma.
\end{aligned}$$

□

2.5 Numerical approximations

To make the bounds from Lemma 14 implementable in software, we just need some simple estimates on α, δ, γ . Using a computer algebra system one may show

Lemma 15. $\alpha \leq 2.879$, $\delta \geq 4.563$, and $\gamma \leq 20.633$.

For instance, the following code in **Maxima** suffices:

```

K : 3.3957;
h : (1+z^2)/(z*(1-z^2)); H : h/K;
g : (1+z^2)/(2*z^3); G:g/K;
gt : (1+z^2)^2/(2*z^3*(3-z^2)); Gt : gt/K;
hh : factor(ratsimp(derivative(h,z)));
F : partfrac(ratsimp(hh/(h+g) -1/(1-z)),z);
Ft : partfrac(ratsimp(hh/(h-gt)-1/(1-z)),z);
assume(z>sqrt(1/3.0)); assume(z<1.0);

```

```

Phi : integrate(ev(F ,z=w),w,1,z);
Phit : integrate(ev(Ft,z=w),w,1,z);
f : K*(1-z)*exp(-Phi);
ft : K*(1-z)*exp(-Phit);
lbintegrand : partfrac(ratsimp(hh/(h*(h-gt))));
t : lbintegrand;
beta : rhs(realroots(Ft)[4]);
alpha : bfloat( 2 * exp( ev(Phi,z=sqrt(1/3.0))
                        -ev(Phit,z=beta)
                        )
              / (%pi * ev(t,z=sqrt(1/3.0)) ) );
delta : bfloat( (2 * %pi)^2 * exp(ev(Phit,z=beta))
                * ev(t,z=sqrt(1/3.0)) / 4);
gamma : bfloat( (2 * %pi)^2
                * exp(ev(Phi,z=sqrt(1/3.0))) / 4);

```

The reader running this code is reminded that **Maxima** displays big-floats in scientific notation with, e.g, `1.0b1` denoting 10, instead of `1.0e1`. The `realroots` command is based on Sturm sequences. Sturm's theorem applies because the numerator of \hat{F} is a univariate polynomial over \mathbf{Z} .

Proof of Theorem 4. The if-direction is plain. For the only-if direction, suppose N is a Dehn filling of M . Then either³ N is a Dehn filling of M along a slope c with $\hat{L}(c) \leq 7.5832$ or N is Dehn filling of M along a slope c with $\hat{L}(c) > 7.5832$. The former case is the first disjunct in Theorem 4. In the latter case, Theorem 5 applies. So ΔV is in the domain of BL and BU , and Lemma 14 applies. Therefore, by equations 7 and 15, the core geodesic of the filling has length ℓ satisfying $\ell < 2.879 \cdot \Delta V$. So N has a geodesic satisfying the first conjunct of the second disjunct of Theorem 4. Furthermore, by equations 8 and 16, $4.563/\Delta V \leq \hat{L}(c)^2$, and by equations 9 and 17, $\hat{L}(c)^2 \leq 20.633/\Delta V$. This is the second and last conjunct of the second and last disjunct of Theorem 4. \square

3 Prospects

The above bounds are all ready to be implemented in code, and finish the theoretical work necessary for a Dehn parental test, modulo the estimates mentioned before—to wit, estimates on volume and normalized length. M. Trnkova, N. Hoffman, the author, and M. Görner are working on implementing such estimates in code.

Once the Dehn parental test is finished, one will be able to calculate the complexity of 3-manifolds for certain notions of complexity, among which is Gabai, Meyerhoff, and Milley's Mom-number m (see [3]). m has the following nice properties:

- if M is a Dehn filling of N , then $m(M) \leq m(N)$;
- for $0 < B < \infty$, there is a finite set $S_m(B)$ such that if $m(N) < B$, then N is a Dehn filling of some element of S_m ; and
- $S_m : [0, \infty) \rightarrow \mathcal{T}$ has an implementation in Regina.

³This can be made constructive and feasible to code by testing $\hat{L}(c) < 7.5832 + \epsilon$ for some reasonably small $\epsilon > 0$.

The reader will note that the volume function v (by Theorems 3.4 and 3.5 of [8]) is known to satisfy these properties as well, except for the last property. Therefore we propose the following natural challenge:

Challenge 16. *To implement, in one’s 3-manifold software suite of choice, a function S_v which, given as input the number 4, runs for at most one week on a 2Ghz processor, and such that for all B , the Dehn fillings of $S_v(B)$ include all orientable hyperbolic 3-manifolds of volume at most B .*

A solution to this challenge would be a significant step towards a proper formulation and eventual solution of the hyperbolic complexity conjecture (see [3]).

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